The number of irreducible components of a linear free divisor

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Outline

Logarithmic vector fields

Let $(D, 0)$ be a hypersurface germ in $(\mathbb{C}^n, 0)$ defined by $f \in \mathscr{O}_{\mathbb{C}^n,0}$.

Definition

The $\mathscr{O}_{\mathbb{C}^n,0}$–module of *logarithmic vector fields* is

$$\begin{aligned} \mathrm{Der}_{\mathbb{C}^n,0}(-\log D) &:= \{\eta \in \mathrm{Der}_{\mathbb{C}^n,0} : \eta(f) \in (f)\} \\ &= \text{the (ambient) vector fields tangent to } D. \end{aligned}$$

Also a Lie algebra.

Question

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How is the structure of D reflected in $\mathrm{Der}_{\mathbb{C}^n,0}(-\log D)$*?*

- *Number of components?*
- *Degrees of defining equations (if homogeneous)?*

Free divisors

Definition (Saito)

$(D,0) \subset (\mathbb{C}^n,0)$ is a *free divisor* if $\mathrm{Der}_{\mathbb{C}^n,0}(-\log D)$ is a free $\mathscr{O}_{\mathbb{C}^n,0}$-module, necessarily of rank n.

$\iff$ $\mathrm{Der}_{\mathbb{C}^n,0}(-\log D)$ is generated by only n elements.

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Examples

Example (Free hyperplane arrangements)

$D = V(xy(x+y)) \subset \mathbb{C}^2$

$$\eta_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad \eta_2 = y(x+y)\frac{\partial}{\partial y}$$

Linear free divisors

Call $(D, 0)$ a *linear free divisor* if $\mathrm{Der}_{\mathbb{C}^n,0}(-\log D)$ has a free basis of *linear* vector fields.

Example (Linear free divisors)

$D = V(x(xz - y^2)) \subset \mathbb{C}^3$

$$\xi_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$
$$\xi_2 = y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$$
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The "opposite" of a free hyperplane arrangement?

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Linear free divisors

Example (Classification of LFDs in $\mathbb{C}^4$, [GMNRS09, §6])

- $xyzw = 0$
  (the *normal crossings divisor*)
- $x(xz - y^2)w = 0$
- $xy(xw - yz) = 0$
- $x(y^3 - 3xyz + 3x^2w) = 0$
- $y^2z^2 - 4xz^3 - 4y^3w + 18xyzw - 27x^2w^2 = 0$
  (the discriminant of the “binary cubics”)

Prehomogeneous vector spaces

Definition

A *prehomogeneous vector space (PVS)* is a rational representation $\rho : G \to \mathrm{GL}(V)$ of a connected complex algebraic group G such that G has a (unique, Zariski) open orbit $\Omega \subset V$.

Differentiating gives a Lie algebra (anti-)homomorphism

$$\tau : \mathfrak{g} \to \mathrm{Der}_{\mathbb{C}^n,0}(-\log(V \setminus \Omega))_0$$

where

$$\mathrm{Der}_{\mathbb{C}^n,0}(-\log(V \setminus \Omega))_0 := \text{linear logarithmic vfs}$$

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Linear free divisors and PVSs

Lemma ([GMNRS09, Lemma 2.3])

Every linear free divisor $(D, 0) \subset V$ *comes from a PVS* $\rho : G \to \mathrm{GL}(V)$ *with* $\dim(G) = \dim(V)$ *and* $D = V \setminus \Omega$.

For such a PVS, say *G defines D*. Then

$$\tau : \mathfrak{g} \to \mathrm{Der}_V(-\log D)_0$$

is an isomorphism.

Facts about Prehomogeneous vector spaces

Let $\rho : G \to \mathrm{GL}(V)$ be a PVS.

Definition

For f a rational function on V and character $\chi : G \to \mathbb{G}_m := \mathrm{GL}(\mathbb{C})$ of G, write '$f \stackrel{\mathrm{m}}{\longleftrightarrow} \chi$' when

$$f(\rho(g)(v)) = \chi(g) \cdot f(v)$$

for all $g \in G$, $v \in \Omega$.

- Let $v_0 \in \Omega$. Then the hypersurface components of $V \setminus \Omega$ are related to the characters of $H = G/[G, G] \cdot G_{v_0}$:

$$\text{number of components} = \mathrm{rank}(\text{character group}(H))$$

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The Main Theorem

Let G define a linear free divisor $(D,0) \subset V$ with components defined by $f_1, \ldots, f_r$, $f_i \overset{\mathrm{m}}{\longleftrightarrow} \chi_i$. Let $v_0 \in \Omega$ and $\mathbb{G}_m := \mathrm{GL}(\mathbb{C})$.

Theorem (P.)

Then $(\chi_1, \ldots, \chi_r) : G \to (\mathbb{G}_m)^r$ *is surjective, with kernel* $H = [G, G] \cdot G_{v_0}$.

Consequences

Theorem (P.)

Then

$$r = \dim(G/[G,G]) = \dim(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]).$$

Also see how elements of generic isotropy subgroups on D permute the level sets of the f_i

Consequences

For the Levi decomposition $G = \mathrm{Rad}_u(G) \rtimes L$, L reductive,

- $r = \dim(\mathrm{Z}(L))$
- $[G, G] = \mathrm{Rad}_u(G) \rtimes [L, L]$
- Homotopy groups of $\Omega = V \setminus D$ reduce to those of $[L, L]$

Examples

Let G define the linear free divisor $(D, 0) \subset V$.

Example ([GMS11, Theorem 2.12])

G is abelian iff D is equivalent to the normal crossings divisor.

Examples

Let G define the linear free divisor $(D, 0) \subset V$

Example ([GMS11, Lemma 2.6])

If G is reductive, then the number of components of D is $\dim(\mathrm{Z}(G))$.

Examples

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Example (P.)

If G is solvable, then the number of components of D is the dimension of a maximal torus of G.

Remarks

- The abstract Lie algebra structure $\mathfrak{g}$ determines the number of components, but **does not** determine the degrees of the functions defining the components!
- More generally, how to do

  $$\mathrm{Der}_{\mathbb{C}^n,0}(-\log D) \mapsto \text{number of components?}$$

  (see also [HM93])
- Questions?

References

[GMNRS09] Michel Granger, David Mond, Alicia Nieto-Reyes, and Mathias Schulze, *Linear free divisors and the global logarithmic comparison theorem*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 2, 811–850. MR 2521436 (2010g:32047)

[GMS11] Michel Granger, David Mond, and Mathias Schulze, *Free divisors in prehomogeneous vector spaces*, Proc. Lond. Math. Soc. (3) **102** (2011), no. 5, 923–950. MR 2795728

[HM93] Herwig Hauser and Gerd Müller, *Affine varieties and Lie algebras of vector fields*, Manuscripta Math. **80** (1993), no. 3, 309–337. MR 1240653 (94j:17025)

[Pik] Brian Pike, *Additive relative invariants and the components of a linear free divisor*, arXiv:1401.2976 [math.RT].

Example

In $\mathbb{C}^5$, the groups defining

$$D_1 : \quad (x_3x_5 - x_4^2)\begin{vmatrix} 0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} = 0,$$

and $D_2 : \quad (x_2^2x_3^2 - 4x_1x_3^3 - 4x_2^3x_4 + 18x_1x_2x_3x_4 - 27x_4^2x_1^2)x_5 = 0$

both have abstract Lie algebra structure $\mathfrak{gl}_2(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C})$