RESEARCH STATEMENT

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1. Introduction

My research concerns the singularities of complex analytic varieties, the singularities of holomorphic maps, and connections between these topics and representation theory. Of particular interest are the 'logarithmic vector fields' and a class of hypersurfaces called 'free divisors'.

Let X be a smooth complex manifold with $p \in X$, and let $\mathscr{O}_{X,p}$ denote the ring of germs at p of holomorphic functions on X. Let $(\mathscr{V}, p)$ be a reduced analytic germ in X, with $\mathscr{V}$ defined by the vanishing of the functions in an $\mathscr{O}_{X,p}$–ideal I. Let $\mathrm{Der}_{X,p}$ denote the germs at p of holomorphic vector fields on X.

Consider the germs of vector fields on the ambient space X that are tangent to $\mathscr{V}$; these *logarithmic vector fields* are formally defined by the condition

$$\mathrm{Der}_{X,p}(-\log \mathscr{V}) := \{\eta \in \mathrm{Der}_{X,p} : \eta(I) \subseteq I\}.$$

This $\mathscr{O}_{X,p}$–module is also an infinite-dimensional Lie algebra, closed under the Lie bracket of vector fields, and may be thought of as the Lie algebra of the group of biholomorphic diffeomorphisms of (X, p) that leave the set $\mathscr{V}$ invariant. It is thus natural to expect the algebraic properties of the logarithmic vector fields to strongly reflect the algebraic and geometric properties of $(\mathscr{V}, p)$.

The module $\mathrm{Der}_{X,p}(-\log \mathscr{V})$ always requires $\geq \dim(X)$ generators. A nonempty germ $(\mathscr{V}, p) \neq (X, p)$ is called a *free divisor* if $\mathrm{Der}_{X,p}(-\log \mathscr{V})$ requires only $\dim(X)$ generators, or equivalently, if $\mathrm{Der}_{X,p}(-\log \mathscr{V})$ is a free $\mathscr{O}_{X,p}$–module, necessarily of rank equal to $\dim(X)$. Geometrically, a free divisor $(\mathscr{V}, p)$ is a hypersurface germ that is either smooth, or maximally singular in the sense that the singular locus $\mathrm{Sing}(\mathscr{V})$ has codimension 1 in $\mathscr{V}$. There is also an algebraic characterization of free divisors due to Aleksandrov [Ale90].

Examples of free divisors include the *free hyperplane arrangements*, where $\mathscr{V}$ is a union of hyperplanes; all reduced plane curve singularities; and all

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discriminants[1] of versal unfoldings of isolated hypersurface and isolated complete intersection singularities, e.g.:

$$xyz(x-y)(x-z)=0, \qquad x^2-y^3=0, \qquad \text{or } b^2-4c=0.$$

Despite being studied since 1980 ([Sai80]), free divisors remain mysterious. For instance, it is not completely understood which hyperplane arrangements are free.

2. Research Objectives

My research aims to understand singular analytic varieties and their associated structures, such as their modules of logarithmic vector fields or the topology of their complement. Free divisors are convenient test subjects: they are numerous, nontrivial but accessible, and they have nice algebraic properties. I particularly enjoy how this work connects with diverse areas of mathematics.

2.1. **Linear free divisors and prehomogeneous vector spaces.** Free divisors classically arose as various types of discriminants, but also have connections to representation theory and harmonic analysis through the study of 'prehomogeneous vector spaces' (see [Sat90]). A hypersurface V in a vector space W is called a *linear free divisor* if $\mathrm{Der}_W(-\log V)$ has a free basis consisting of *linear vector fields* such as $2x\partial_x - z\partial_y$ or $(x-y)\partial_y$. If V is a linear free divisor, then it is defined by a homogeneous polynomial of degree $\dim(W)$.

Each linear free divisor arises from a *prehomogeneous vector space*, a rational representation $\rho : G \to \mathrm{GL}(W)$ of a connected complex linear algebraic group G on a vector space W, such that ρ has an open orbit Ω in W. Then Ω is Zariski open, and its complement is an algebraic set. As all $\rho(g)$ leave invariant Ω^c, differentiating ρ gives a Lie algebra (anti-)homomorphism $d\rho_{(e)} : \mathfrak{g} \to \mathrm{Der}_W(-\log \Omega^c)$, where $\mathfrak{g}$ is the Lie algebra of G and the image is a finite-dimensional Lie algebra of linear logarithmic vector fields (e.g., [DP15, §1]).

By [GMNRS09, §2], every linear free divisor $V \subset W$ is of the form Ω^c for a prehomogeneous vector space $\rho : G \to \mathrm{GL}(W)$ with $\dim(G) = \dim(W)$ and a "reduced" Ω^c, and conversely, and then $\mathrm{Der}_W(-\log \Omega^c)$ is generated by $d\rho_{(e)}(\mathfrak{g})$. Among all prehomogeneous vector spaces, those that give linear free divisors are the extremal class for which the group is of minimal possible dimension, and yet the group action generates all logarithmic vector fields.

[1] For example, the singularity defined by $f(x) = x^n$ has $(x, c_1, \ldots, c_n) \mapsto x^n + \sum_i c_i x^{n-i}$ as one possible versal unfolding, and the discriminant of this unfolding is defined by the *classical discriminant* Δ of the generic monic degree n polynomial $x^n + \sum_i c_i x^{n-i}$; recall that Δ is a polynomial in $c_1, \ldots, c_n$ such that $\Delta(c_1, \ldots, c_n) = 0$ iff $x^n + \sum_i c_i x^{n-i}$ has a multiple root.

2.1.1. *Finding linear free divisors.* The initial examples of linear free divisors came primarily from quivers ([GMNRS09, BM06]). More recently, Granger–Mond–Schulze [GMS11] showed that up to 'castling transformation', the irreducible representations that give linear free divisors are known by a classification of certain prehomogeneous vector spaces by Sato–Kimura [SK77]. In these examples, the group is always reductive.

In contrast, Damon and I [DP15] studied linear free divisors coming from representations of solvable linear algebraic groups. These are often arranged as infinite 'towers' of linear free divisors that come from 'towers' of representations of solvable groups. We also gave examples of linear free divisors for which the group is neither solvable nor reductive, and observed a pattern that solvable group extensions often allow new linear free divisors to be constructed from old, in some cases automatically producing an infinite tower from a single linear free divisor (e.g., [Pik10, §5.3]). It remains to thoroughly understand this phenomenon.

2.1.2. *The structure of linear free divisors.* Granger–Mond–Schulze [GMS11] described the structure of linear free divisors with reductive groups; for instance, the number of irreducible components of the divisor equals the dimension of the center of $G \subset \mathrm{GL}(W)$, and the number of irreducible G–modules in W. More recently, I used a criterion of Brion [Bri06, GMS11] to show [Pika] that for a prehomogeneous vector space $\rho : G \to \mathrm{GL}(W)$ defining any linear free divisor V, there are no nontrivial rational representations $G \to (\mathbb{C}, +)$. Then by the theory of prehomogeneous vector spaces developed by Mikio Sato, and some structure theory for linear algebraic groups, the number of irreducible components of V is equal to $\dim(G/[G, G])$; this gives significant insight into the structure of the groups and representations that produce linear free divisors. For instance, the Lie algebra $\mathfrak{g}$ of G is the direct sum of $[\mathfrak{g}, \mathfrak{g}]$ and an abelian subalgebra. Also, the isotropy subgroup at a generic point on an irreducible component $V(f_i)$ of a linear free divisor $V(f_1 \cdots f_k) = \Omega^c$ nontrivially permutes the level sets of f_i and leaves invariant all level sets of all f_j, $j \neq i$.

With further study, this may lead to a structure theorem for linear free divisors. All linear free divisors known to me take the form of a reductive linear free divisor that is then extended by a solvable group in a process described in [DP15], to produce a 'mixed' linear free divisor on a larger space.

2.1.3. *Classifying linear free divisors.* Any conjecture on the structure of linear free divisors must be informed by a variety of examples. Although [GMNRS09] classified linear free divisors in $\mathbb{C}^k$ for $k \leq 4$, these examples are not sufficiently complicated; for example, they are all either solvable or reductive, never mixed. In 2011, I attempted to classify the linear free divisors in $\mathbb{C}^5$. Though this project has not yet been finished, it produced interesting new examples and led to the results in [Pika]. Brent Pym [Pym13] also used these examples to produce new examples of Poisson structures.

2.1.4. *Deforming linear free divisors.* Torielli [Tor12] studied deformations of linear free divisors, and proved that reductive linear free divisors are rigid and cannot be deformed to an inequivalent linear free divisor. Although he conjectured that this was true for all linear free divisors, my classification work produced a solvable counterexample in $\mathbb{C}^5$. In addition to furthering his work, it would be interesting to study the deformations of prehomogeneous vector spaces. For example, the variety V of singular 4×4 skew-symmetric matrices is a component of the complement of the open orbit of a particular prehomogeneous vector space ρ, but there is no known linear free divisor that contains V as a component; may ρ be deformed to produce such a linear free divisor?

2.1.5. *The topology of the complement of a linear free divisor.* One way to study a hypersurface is to study the topology of its complement. For free divisors specifically, there are at least two problems of interest. The first is to determine when the complement of a free divisor is an Eilenberg–MacLane space of type $K(\pi, 1)$, that is, with trivial nth homotopy group for $n > 1$. For instance, this includes a conjecture of Saito regarding free hyperplane arrangements (see [OT92]), as well as the classical "$K(\pi, 1)$ problem" for versal deformations of isolated hypersurface singularities. The second is to determine when the *logarithmic comparison theorem* holds, that is, when the cohomology of the complement may be computed by the complex of *logarithmic differential forms* that have controlled poles along the divisor (e.g., [CJNMM96, GMNRS09]).

Since the complement of a linear free divisor is diffeomorphic to G/G_{v_0} for G_{v_0} a discrete isotropy subgroup, these problems are more accessible for linear free divisors. For instance, Damon and I [DP12] showed that for a large class of examples of solvable linear free divisors, both the complements of the linear free divisors and the Milnor fiber of their defining equations are $K(\pi, 1)$'s. In [Pika], I showed that for $n > 1$, the nth homotopy group of the complement of a linear free divisor is equal to the nth homotopy group of the semisimple part of a Levi subgroup of G. In future work, [Pika] should also provide insight into the logarithmic comparison theorem problem.

2.2. **The ubiquity of free divisors.** Linear free divisors are simplified test cases for many questions about arbitrary free divisors, and indeed that is the origin of many of the questions in §2.1. There are other questions specific to free divisors.

2.2.1. *Discriminants of deformations.* A recurring pattern is that the discriminant of a suitable deformation of a certain type of singularity is a free divisor. For instance, free divisors classically arose as discriminants of versal unfoldings of isolated hypersurface and isolated complete intersection singularities. Damon [Dam98] gave sufficient conditions for a group of equivalences $\mathscr{G}$ to have the property that the discriminant of any $\mathscr{G}$-versal unfolding is a free divisor. A linear free divisor is in some sense the discriminant of a prehomogeneous vector space. Buchweitz has suggested that "every free divisor is the discriminant of something," and it would be very interesting to construct, from a free divisor $\mathscr{V}$, a deformation with discriminant equal to $\mathscr{V}$.

2.2.2. *Pulling back free divisors.* A natural question is to understand the behavior of free divisors under various operations. If $\varphi : X \to Y$ is a map between smooth spaces, and $\mathscr{V}$ is a free divisor in Y, when is $\varphi^{-1}(\mathscr{V})$ a free divisor? Buchweitz and I [BP] showed that when all $\eta \in \mathrm{Der}_Y(-\log \mathscr{V})$ lift across φ, and the deformation module $T^1_{X/Y}$ of φ is a Cohen–Macaulay $\mathscr{O}_{X,p}$–module of codimension 2, then $\varphi^{-1}(\mathscr{V})$ is also a free divisor. (For instance, we found many instances from invariant theory where these hypotheses are satisfied for the algebraic quotient $\varphi : X \to X//G$, where G is a reductive group and X is a G-representation.) However, our hypotheses may be unnecessarily restrictive: Mond–Schulze [MS13] have identified cases in which pulling back a non-free divisor produces a free divisor. We should determine the exact conditions for $\varphi^{-1}(\mathscr{V})$ to be a free divisor.

Conversely, if $\varphi^{-1}(\mathscr{V})$ is a free divisor, what may be said about $\mathrm{Der}_Y(-\log \mathscr{V})$? One application for this would be in the study of castling transformations. If $M_{p,q}$ denotes the space of $p \times q$ complex matrices and $n > m$, then Granger–Mond–Schulze [GMS11] showed that under the castling operation of prehomogeneous vector spaces, a linear free divisor in $M_{n,n-m}$ transforms to a linear free divisor in $M_{n,m}$, and vice-versa; both linear free divisors are polynomials in the maximal minors of these spaces of matrices, and the castling transformation swaps corresponding maximal minors. To generalize this to arbitrary free divisors, as Buchweitz and I did [BP] in one direction of the $m = 1$ case, it would be very useful to describe $\mathrm{Der}_Y(-\log f)$ when $f \circ \varphi$ defines a free divisor, in particular for $\varphi : M_{n,m} \to \mathbb{C}^{\binom{n}{m}}$ that evaluates all maximal minors.

2.2.3. *Free completions.* Let $(\mathscr{V}_1, p)$ be a reduced hypersurface in (X, p). It is natural to ask whether every such hypersurface is part of some free divisor $(\mathscr{V}_1 \cup \mathscr{V}_2, p)$, which is called a *free completion* of $\mathscr{V}_1$; sometimes, $\mathscr{V}_2$ is required to itself be a free divisor. Since $\mathrm{Der}_X(-\log(\mathscr{V}_1 \cup \mathscr{V}_2)) = \mathrm{Der}_X(-\log \mathscr{V}_1) \cap \mathrm{Der}_X(-\log \mathscr{V}_2)$, this is a question about how modules of logarithmic vector fields may intersect. Mond–Schulze [MS13] have found instances of free completions using data from the normalization of $\mathscr{V}_1$. We should describe those hypersurfaces that have a free completion, and be able to construct the free completions.

2.3. **The structure of the logarithmic vector fields.** For a particular $\mathscr{V} \subset X$ defined by an ideal I, $\mathrm{Der}_{X,p}(-\log \mathscr{V})$ may be readily computed by a computer as certain syzygies: for instance, a relation $\sum_i a_i \frac{\partial f}{\partial x_i} + bf = 0$ corresponds to $\sum_i a_i \frac{\partial}{\partial x_i} \in \mathrm{Der}_X(-\log V(f))$. However, many natural situations produce a submodule $L \subseteq \mathrm{Der}_{X,p}(-\log \mathscr{V})$, and an open problem is to find necessary and sufficient conditions for $L = \mathrm{Der}_{X,p}(-\log \mathscr{V})$. For free divisors, this is accomplished by a criterion of Saito [Sai80] where the condition is that the determinant of a presentation matrix M of L must be reduced in $\mathscr{O}_{X,p}$.

2.3.1. *Fitting ideals.* For an arbitrary $(\mathscr{V}, p)$, such a presentation matrix M is not square, and hence it is natural to instead consider the ideals generated by minors of M of a particular size, the *Fitting ideals* of $\mathrm{Der}_{X,p}/L$. In [Pikb] I found upper bounds for these Fitting ideals, and gave a geometric interpretation to these ideals. I also showed that for an arbitrary hypersurface $\mathscr{V}$, $L = \mathrm{Der}_{X,p}(-\log \mathscr{V})$ if and only if both the 0th Fitting ideal of $\mathrm{Der}_{X,p}/L$ is in some sense "reduced" with respect to all components of $\mathscr{V}$, and L is a reflexive module. (If L is free, then L is also reflexive and this simplifies to Saito's criterion.)

A simple example shows that for non-hypersurfaces, the Fitting ideals alone are insufficient to prove $L = \mathrm{Der}_{X,p}(-\log \mathscr{V})$. In future work, I will investigate whether some generalization of reflexivity holds for $\mathrm{Der}_{X,p}(-\log \mathscr{V})$ when $\mathrm{codim}(\mathscr{V}) > 1$; this may be the key to generalizing Saito's work further.

2.3.2. *Lie algebra structure.* Another avenue for answering this question lies in the work of Hauser–Müller [HM93]. They investigate the Lie algebra structure of the logarithmic vector fields, and show that modules of the form $\cap_{i=1}^{k} \mathrm{Der}_{X,p}(-\log V_i)$ are distinguished by being the end of a chain of subalgebras having a "maximal balanced" property. It would be very interesting to find a practical means to check this property, and moreover to check if $k = 1$.

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