Outline

Contents

1 Free divisors

1.1 Free divisors

Free divisors

Let θ_n be germs of holomorphic vector fields at 0 in $\mathbb{C}^n$.

For hypersurface $(V,0) \subset (\mathbb{C}^n, 0)$, define the $\mathscr{O}_{\mathbb{C}^n,0}$-module and Lie algebra

$$\mathrm{Derlog}(V) = \{\eta \in \theta_n | \eta(f) \in I(V) \text{ for all } f \in I(V)\}.$$

Definition 1 (Saito)**.** If $\mathrm{Derlog}(V)$ is free (of rank n), then $(V,0)$ is a *free divisor*.

Our Problem

- For a vector space of square matrices (e.g., $\mathrm{Sym}_n(\mathbb{C})$, $M(n,\mathbb{C})$, $\mathrm{Sk}_{2k}(\mathbb{C})$), find a free divisor V which includes the hypersurface of singular matrices as a component.
- Even better if $V = H^{-1}(0)$ is *H-holonomic*

1.2 Free divisors from representations

Saito's criterion

Theorem 2 (Saito)**.** *Let $\delta^1, \cdots, \delta^n \in \theta_n$ with $\delta^i = \sum_{j=1}^n a_{ji}(z)\frac{\partial}{\partial z_j}$. Let $M = \mathscr{O}_{\mathbb{C}^n,0}\{\delta^1, \ldots, \delta^n\}$. If*

1. *M is a Lie algebra, and*
2. *the* coefficient determinant $h = \det(a_{ji}(z))$ *defines a reduced hypersurface* $(V, 0)$,

then $(V, 0)$ *is a free divisor with* $\mathrm{Derlog}(V) = M$.

Representations

Definition 3. An *equidimensional representation* $\rho : G \to \mathrm{GL}(\mathscr{W})$ is a rational representation of a connected complex algebraic Lie group with an open orbit Ω and $n = \dim_{\mathbb{C}}(G) = \dim_{\mathbb{C}}(\mathscr{W})$.

- If $E_1, \ldots, E_n \in \mathfrak{g}$ is a basis and each $\delta^i = \xi_{E_i}$ is a vector field on $\mathscr{W}$ obtained by differentiating ρ, then $h = \det(a_{ji}(z))$ defines the *exceptional orbit variety* $V = \mathscr{W} \setminus \Omega$.
- (Mond) By Saito's criterion, if h is reduced then V is a *linear free divisor*.
- If h is not reduced then V has a "free* divisor structure."

Linear free divisors

- Linear free divisors from quiver representations have been studied ([BM06, GMNRS09]); these use reductive groups
- We use solvable groups.

2 Free divisors from solvable groups

2.1 Motivation

Matrix factorizations

The complex analogues of the following matrix factorizations involve equidimensional representations of solvable groups:

- Cholesky factorization for symmetric matrices
- LU factorization for general $n \times n$ matrices
- A Cholesky-like factorization for skew-symmetric matrices

Ex: Cholesky representation

- Let $L_n(\mathbb{C})$ be the group of $n \times n$ invertible lower triangular matrices.
- $L_n(\mathbb{C})$ acts on $\mathrm{Sym}_n(\mathbb{C})$ by $A \cdot M = AMA^T$.
- Equidimensional!
- Exceptional orbit variety?

Cholesky representation

Matrix of coefficients for 3×3, for some choice of bases:

$$\begin{pmatrix} 2x_{11} & 0 & 0 & 0 & 0 & 0 \\ x_{12} & x_{11} & 0 & x_{12} & 0 & 0 \\ x_{13} & 0 & x_{11} & 0 & x_{12} & x_{13} \\ 0 & 2x_{12} & 0 & 2x_{22} & 0 & 0 \\ 0 & x_{13} & x_{12} & x_{23} & x_{22} & x_{23} \\ 0 & 0 & 2x_{13} & 0 & 2x_{23} & 2x_{33} \end{pmatrix}$$

Is determinant reduced?

Cholesky representation

- Consider the partial flag of invariant subspaces of $\mathrm{Sym}_3(\mathbb{C})$:

$$\{0\} \subset \left\{ \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \right\} \subset \mathrm{Sym}_3(\mathbb{C}).$$

- Kernels of corresponding quotient representations ($L_3(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_3(\mathbb{C})/W)$) are

$$\{\pm I\} \subset \left\{ \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & * \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\} \subset L_3(\mathbb{C}).$$

Cholesky representation

With bases chosen to respect this structure:

$$\begin{pmatrix} 2x_{11} & 0 & 0 & 0 & 0 & 0 \\ x_{12} & x_{11} & x_{12} & 0 & 0 & 0 \\ 0 & 2x_{12} & 2x_{22} & 0 & 0 & 0 \\ x_{13} & 0 & 0 & x_{11} & x_{12} & x_{13} \\ 0 & x_{13} & x_{23} & x_{12} & x_{22} & x_{23} \\ 0 & 0 & 0 & 2x_{13} & 2x_{23} & 2x_{33} \end{pmatrix}$$

Exceptional orbit variety is the free divisor defined by

$$x_{11} \cdot \begin{vmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{vmatrix} \cdot \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{vmatrix} = 0.$$

2.2 Block representations

Block representations

Let $\rho : G \to \mathrm{GL}(W)$ be a rational representation of a connected complex algebraic Lie group with a partial flag of invariant subspaces

$$\{0\} = W_0 \subset \cdots \subset W_l = W.$$

Let $K_j = \ker(G \to \mathrm{GL}(W/W_j))$, so that

$$K_0 \subset \cdots \subset K_l = G.$$

Block representations

Definition 4. ρ (with the invariant subspaces) is a *candidate block representation* if

1. $\dim_{\mathbb{C}}(K_j) = \dim_{\mathbb{C}}(W_j)$ for $j = 1, \ldots, l$, and
2. the *relative coefficient determinant* $g_j : W \to \mathbb{C}$ is nonzero for $j = 1, \ldots, l$.

If also

3. each g_j is reduced and $\{g_j\}$ are relatively prime,

then ρ is a *block representation*. If (3) does not hold, ρ is a *non-reduced block representation*.

The matrix of a block representation

Using bases complementary to W_j in W_{j+1} and $\mathfrak{k}_j$ in $\mathfrak{k}_{j+1}$ makes the matrix of coefficients block lower triangular:

$$\begin{matrix} W_l/W_{l-1}\{ \\ \\ W_1/W_0 \ \ \{ \end{matrix} \begin{pmatrix} \overbrace{A_{1,1}}^{\mathfrak{k}_l/\mathfrak{k}_{l-1}} & \cdots & \overbrace{0}^{\mathfrak{k}_1/\mathfrak{k}_0} \\ \vdots & \ddots & \vdots \\ A_{l,1} & \cdots & A_{l,l} \end{pmatrix},$$

Note that $g_j = \det(A_{j,j})$.

The exceptional orbit variety of a block representation

Theorem 5. *If ρ is a block representation, then its exceptional orbit variety is a linear free divisor defined by*

$$\prod_{j=1}^{l} g_j = 0. \tag{1}$$

If ρ is a non-reduced block representation, then its exceptional orbit variety is a linear free divisor defined with non-reduced structure by* (1).

2.3 More free divisors

Cholesky factorization

$L_n(\mathbb{C})$ acts on $\mathrm{Sym}_n(\mathbb{C})$ by $A \cdot M = AMA^T$.

- Block representations
- Theorem ([BM06, GMNRS09]): Free divisors for each $n \in \mathbb{N}$
- Theorem ([GMNRS09], Damon-P.): Each of these is H-holonomic.

LU factorization

Let $G = L_n(\mathbb{C}) \times$ (unipotent $n \times n$ upper triangular matrices) act on $M(n,n,\mathbb{C})$ by $(A,B) \cdot M = AMB^{-1}$.

- Non-reduced block representations
- Free* divisors

Modified LU factorization

Instead, use the group

$$\left\{ \left(A, \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \right) \middle| A \in L_n(\mathbb{C}), B^T \in L_{n-1}(\mathbb{C}) \right\}.$$

- Block representations
- Theorem (Damon-P.): Free divisor for each $n \in \mathbb{N}$
- Theorem (Damon-P.): Each of these is H-holonomic
- Analogous results for the $n \times (n+1)$ matrices

Example: Modified LU factorization

All free divisors are defined by a product of nested determinants.

Example 6. For 3×3 general matrices, the free divisor obtained from the modified LU factorization is defined by

$$x_{11} \cdot \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \cdot \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \cdot x_{12} \cdot \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} = 0$$

Cholesky-like factorization for skew-symmetric

Let $G \subset L_n(\mathbb{C})$ consist of all matrices with 2×2 blocks of the form $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$ down the diagonal. Let G act on $\mathrm{Sk}_n(\mathbb{C})$ by $A \cdot M = AMA^T$.

- Non-reduced block representation
- Free* divisors
- Conjecture: no subgroup of $L_n(\mathbb{C})$ gives a free divisor for $n \geq 4$. $\mathrm{GL}_n(\mathbb{C})$?

Free divisors for skew-symmetric matrices

Apply Saito's criterion directly using

- $\binom{n}{2} - (n-3)$ linear vector fields coming from

  $$\left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & A \end{pmatrix} \middle| \lambda_i \neq 0, A \in L_{n-2}(\mathbb{C}) \right\}$$

  acting on $\mathrm{Sk}_n(\mathbb{C})$ by $A \cdot M = AMA^T$

- and $n-3$ nonlinear Pfaffian vector fields of the form

$$\eta_{ab} = \sum_{b<p<q\leq n} S(a\cdots bpq)\frac{\partial}{\partial x_{pq}},$$

  where each coefficient is a particular Pfaffian.

Theorem (Damon-P.): The module these vector fields generate is an infinite-dimensional "solvable" Lie algebra.

Free divisors for skew-symmetric matrices

Theorem (Damon-P.): We obtain free divisors on $\mathrm{Sk}_n(\mathbb{C})$ for all $n \geq 3$.

Example 7. When $n = 4$, coefficient matrix is

$$\begin{pmatrix} x_{12} & x_{12} & 0 & 0 & 0 & 0 \\ x_{13} & 0 & x_{13} & 0 & 0 & 0 \\ 0 & x_{23} & x_{23} & 0 & 0 & 0 \\ x_{14} & 0 & 0 & x_{13} & x_{14} & 0 \\ 0 & x_{24} & 0 & x_{23} & x_{24} & 0 \\ 0 & 0 & x_{34} & 0 & x_{34} & \mathrm{Pf} \end{pmatrix}$$

Summary of free divisors

- Linear free divisors:
  - Symmetric $n \times n$, for all $n \in \mathbb{N}$ (Cholesky)
  - General $n \times n$, for all $n \in \mathbb{N}$ (Modified LU)
  - General $n \times (n+1)$, for all $n \in \mathbb{N}$ (Modified LU)
- Skew-symmetric $n \times n$, for all $n \geq 3$ (not linear)

3 Extensions of linear free divisors

3.1 Results

Extension results

- Symmetric:
  - Say the restriction of

    $$\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C})), \qquad A \cdot M = AMA^T$$

    to a subgroup and subspace gives a linear free divisor.
  - We give sufficient conditions that a "solvable" extension to the $(n+1)\times(n+1)$ case gives a linear free divisor.
- Similar results for the general $n \times m$ matrices with action $(A, B) \cdot M = AMB^{-1}$.

3.2 Examples

Extension example 1 (symmetric)

Example 8. • The diagonal invertible $m \times m$ matrices acting on the diagonal $m \times m$ matrices by $A \cdot M = AMA^T$ gives a "normal crossings" linear free divisor defined by

$$\prod_{i=1}^{m} x_{i,i} = 0.$$

Extension example 1 (symmetric)

Example 9. • Can extend to to a free divisor defined (with non-reduced structure) by the product of the determinants of the upper left square submatrices of

$$\begin{pmatrix} x_{1,1} & & & x_{1,m+1} & \cdots & x_{1,n} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & x_{m,m} & x_{m,m+1} & \cdots & x_{m,n} \\ x_{1,m+1} & \cdots & x_{m,m+1} & x_{m+1,m+1} & \cdots & x_{m+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & \cdots & x_{m,n} & x_{m+1,n} & \cdots & x_{n,n} \end{pmatrix}.$$

Extension example 2 (symmetric)

A non-solvable example:

Example 10. • The group $\left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}) \right\}$ acting on $\left\{ \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \in \mathrm{Sym}_3(\mathbb{C}) \right\}$ gives a linear free divisor defined by

$$\begin{vmatrix} x_{22} & x_{23} \\ x_{23} & x_{33} \end{vmatrix} \cdot \begin{vmatrix} 0 & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = 0.$$

Extension example 2 (symmetric)

A non-solvable example:

Example 11. • Solvable extensions add "generic determinants" to give linear free divisors on

$$\{A \in \mathrm{Sym}_n(\mathbb{C}) | (A)_{1,1} = 0\}$$

for all $n \geq 3$.

3.3 Future work

Future work

- When may we just "change the group" to obtain a linear free divisor?
- Is there a general extension mechanism for linear free divisors?
- When is an extension H-holonomic?
- Why do many free divisors take the form of "determinantal arrangements"?
- Questions?

References

References

[BM06] Ragnar-Olaf Buchweitz and David Mond, *Linear free divisors and quiver representations*, Singularities and computer algebra, London Math. Soc. Lecture Note Ser., vol. 324, Cambridge Univ. Press, Cambridge, 2006, pp. 41–77. MR 2228227 (2007d:16028)

[GMNRS09] Michel Granger, David Mond, Alicia Nieto-Reyes, and Mathias Schulze, *Linear free divisors and the global logarithmic comparison theorem*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 2, 811–850. MR 2521436 (2010g:32047)

[Sai80] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291. MR 586450 (83h:32023)