Block representations and their properties

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Outline

Motivation

- For a vector space of square matrices (e.g., $\mathrm{Sym}_n(\mathbb{C})$, $M(n, \mathbb{C})$, $\mathrm{Sk}_{2k}(\mathbb{C})$), find a free divisor V which includes the hypersurface of singular matrices as a component.
- Even better if
  - $V = (\text{free divisor}) \cup \{\text{singular matrices}\}$
  - $V = H^{-1}(0)$ is *H-holonomic*

Saito's criterion

Theorem (Saito)

Let $\delta^1, \cdots, \delta^n \in \theta_n$ with $\delta^i = \sum_{j=1}^n a_{ji}(z)\frac{\partial}{\partial z_j}$. Let $M = \mathscr{O}_{\mathbb{C}^n,0}\{\delta^1, \ldots, \delta^n\}$. If

1. *M is a Lie algebra, and*
2. *the coefficient determinant $h = \det(a_{ji}(z))$ defines a reduced hypersurface $(V, 0)$,*

then $(V, 0)$ is a free divisor with $\mathrm{Derlog}(V) = M$.

Representations

Definition

An *equidimensional representation* $\rho : G \to \mathrm{GL}(\mathscr{W})$ is a rational representation of a connected complex algebraic Lie group with an open orbit Ω and $n = \dim_{\mathbb{C}}(G) = \dim_{\mathbb{C}}(\mathscr{W})$.

- If $E_1, \ldots, E_n \in \mathfrak{g}$ is a basis and each $\delta^i = \xi_{E_i}$ is a vector field on $\mathscr{W}$ obtained by differentiating ρ, then $h = \det(a_{ji}(z))$ defines the *exceptional orbit variety* $V = \mathscr{W} \setminus \Omega$.
- (Mond) By Saito's criterion, if h is reduced then V is a *linear free divisor*.
- If h is not reduced then V has a "free* divisor structure.

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Linear free divisors

- Linear free divisors using primarily reductive groups have been studied (e.g., [BM06], [GMNRS09], etc.)
- We use primarily solvable groups.

Why solvable groups?

1. The complex analogues of the following matrix factorizations involve equidimensional representations of solvable groups:
   - Cholesky factorization for symmetric matrices
   - LU factorization for general $n \times n$ matrices
   - A Cholesky-like factorization for skew-symmetric matrices
2. Representations of solvable groups have a complete flag of invariant subspaces (Lie-Kolchin Theorem)

Ex: Cholesky representation

- Let $L_n(\mathbb{C})$ be the group of $n \times n$ invertible lower triangular matrices.
- $L_n(\mathbb{C})$ acts on $\mathrm{Sym}_n(\mathbb{C})$ by $A \cdot M = AMA^T$.
- Equidimensional!
- Exceptional orbit variety?

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Ex: Cholesky representation

Matrix of coefficients for 3×3, for some choice of bases:

$$\begin{pmatrix} 2x_{11} & 0 & 0 & 0 & 0 & 0 \\ x_{12} & x_{11} & 0 & x_{12} & 0 & 0 \\ x_{13} & 0 & x_{11} & 0 & x_{12} & x_{13} \\ 0 & 2x_{12} & 0 & 2x_{22} & 0 & 0 \\ 0 & x_{13} & x_{12} & x_{23} & x_{22} & x_{23} \\ 0 & 0 & 2x_{13} & 0 & 2x_{23} & 2x_{33} \end{pmatrix}$$

Is determinant reduced?

Ex: Cholesky representation

- Consider the partial flag of invariant subspaces of $\mathrm{Sym}_3(\mathbb{C})$:

$$\{0\} \subset \left\{ \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \right\} \subset \mathrm{Sym}_3(\mathbb{C}).$$

- Kernels of corresponding quotient representations $(L_3(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_3(\mathbb{C})/W))$ are

$$\{\pm I\} \subset \left\{ \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & * \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\} \subset L_3(\mathbb{C}).$$

Ex: Cholesky representation

- Consider the partial flag of invariant subspaces of $\mathrm{Sym}_3(\mathbb{C})$:

$$\{0\} \subset \left\{ \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \right\} \subset \mathrm{Sym}_3(\mathbb{C}).$$

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Ex: Cholesky representation

With bases chosen to respect this structure:

$$\begin{pmatrix} 2x_{11} & 0 & 0 & 0 & 0 & 0 \\ x_{12} & x_{11} & x_{12} & 0 & 0 & 0 \\ 0 & 2x_{12} & 2x_{22} & 0 & 0 & 0 \\ x_{13} & 0 & 0 & x_{11} & x_{12} & x_{13} \\ 0 & x_{13} & x_{23} & x_{12} & x_{22} & x_{23} \\ 0 & 0 & 0 & 2x_{13} & 2x_{23} & 2x_{33} \end{pmatrix}$$

Exceptional orbit variety is the free divisor defined by

$$x_{11} \cdot \begin{vmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{vmatrix} \cdot \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{vmatrix} = 0.$$

Block representations

Let $\rho : G \to \mathrm{GL}(W)$ be a rational representation of a connected complex algebraic Lie group with a partial flag of invariant subspaces

$$\{0\} = W_0 \subset \cdots \subset W_l = W.$$

Let $K_j = \ker(G \to \mathrm{GL}(W/W_j))$, so that

$$K_0 \subset \cdots \subset K_l = G.$$

Block representations

Definition

ρ (with the invariant subspaces) is a *candidate block representation* if

1. $\dim_{\mathbb{C}}(K_j) = \dim_{\mathbb{C}}(W_j)$ for $j = 1, \ldots, l$, and
2. the *relative coefficient determinant* $g_j : W \to \mathbb{C}$ is nonzero for $j = 1, \ldots, l$.

If also

3. each g_j is reduced and $\{g_j\}$ are relatively prime,

then ρ is a *block representation*. If (3) does not hold, ρ is a *non-reduced block representation*.

The matrix of a block representation

Using bases complementary to W_j in W_{j+1} and $\mathfrak{k}_j$ in $\mathfrak{k}_{j+1}$ makes the matrix of coefficients block lower triangular:

$$\begin{matrix} W_l/W_{l-1}\{ \\ \\ W_1/W_0\ \{ \end{matrix}\begin{pmatrix} \overbrace{A_{l,l}}^{\mathfrak{k}_l/\mathfrak{k}_{l-1}} & \cdots & \overbrace{0}^{\mathfrak{k}_1/\mathfrak{k}_0} \\ \vdots & \ddots & \vdots \\ A_{l,1} & \cdots & A_{1,1} \end{pmatrix},$$

Note that $g_j = \det(A_{j,j})$.

The exceptional orbit variety of a block representation

Theorem

If ρ is a block representation, then its exceptional orbit variety is a linear free divisor defined by

$$\prod_{j=1}^{l} g_j = 0. \tag{1}$$

If ρ is a non-reduced block representation, then its exceptional orbit variety is a linear free divisor defined with non-reduced structure by* (1).

Quotient Property

If $\rho : G \to \mathrm{GL}(W)$ is a block representation with invariant subspaces

$$\{0\} = W_0 \subset \cdots \subset W_l = W,$$

then $\overline{\rho} : G/K_j \to \mathrm{GL}(W/W_j)$ is a block representation with invariant subspaces

$$\{0\} \simeq W_j/W_j \subset \cdots \subset W_l/W_j = W/W_j$$

and coefficient determinant

$$\prod_{i=j+1}^{l} g_i.$$

Thus have (free divisor) $=$ (free divisor) $\cup$ (another hypersurface).

Quotient Property: Example

From the 3×3 Cholesky representation, can recover

$$x_{11} = 0 \qquad \text{and} \qquad x_{11} \cdot \begin{vmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{vmatrix} = 0$$

using the quotient property.

Extension Property

Let $\rho : G \to \mathrm{GL}(W)$ be a representation, $V \subset W$ an invariant subspace, and $K = \ker(G \to \mathrm{GL}(W/V))$. If

- $\overline{\rho} : G/K \to \mathrm{GL}(W/V)$ is a block representation,
- $\dim(V) = \dim(K)$, and
- the relative coefficient determinant for K acting on V is reducible and relatively prime to the coefficient determinant of $\overline{\rho}$

then ρ is a block representation with invariant subspaces

$$\{0\} \subset V \subset \pi^{-1}(W_1) \subset \cdots \subset \pi^{-1}(W_{l-1}) \subset W$$

and with one new relative coefficient determinant.

Towers

Often have homomorphisms between representations of matrix groups on spaces of matrices (i.e., pad in obvious ways)

A *tower of block representations* is a chain of inclusions of representations

$$(G_1, W_1) \hookrightarrow (G_2, W_2) \hookrightarrow (G_3, W_3) \hookrightarrow \cdots ,$$

each of which is a block representation, such that for all j,

$$(G_{j-1}, W_{j-1}) \hookrightarrow (G_j, W_j) \to (G_j/K_j, W_j/V_j)$$

is an isomorphism (here, V_j is the largest nontrivial invariant subspace in the block representation of (G_j, W_j) and K_j is the corresponding kernel).

Consequence: Add terms to the free divisors at every step

Restriction Property

Let $\rho : G \to \mathrm{GL}(W)$ be a block representation. If

- we have a G-invariant subspace

$$W_{j-1} \subset \overline{W} \subset W_j$$

  and an algebraic group

$$K_{j-1} \subset \overline{K} \subset K_j$$

  with $\dim(\overline{W}) = \dim(\overline{K})$, and
- the relative coefficient determinant of $\overline{K}/K_{j-1}$ on $\overline{W}/W_{j-1}$ and $g_j|_{\overline{W}}$ are reduced and relatively prime,

then restricting ρ to

$$\overline{K} \to \mathrm{GL}(\overline{W})$$

gives a block representation.

Examples on symmetric: 1

$L_1(\mathbb{C})$ acting on $\mathrm{Sym}_1(\mathbb{C})$ by $A \cdot X = AXA^T$ gives LFD

$$x_{11} = 0.$$

Using extension property, get a tower of LFDs

$$x_{11} \cdot \begin{vmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{vmatrix} \cdot \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{vmatrix} \cdot \begin{vmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{12} & x_{22} & x_{23} & x_{24} \\ x_{13} & x_{23} & x_{33} & x_{34} \\ x_{14} & x_{24} & x_{34} & x_{44} \end{vmatrix} \cdots$$

Examples on symmetric: 2

Or, start with normal crossings divisor $\prod_{i=1}^{m} x_{ii} = 0$ (using diagonal matrices for the group and the vector space). Use extension property to get a tower of LFDs defined by principal minors of

$$\begin{pmatrix} x_{1,1} & & & x_{1,m+1} & \cdots & x_{1,n} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & x_{m,m} & x_{m,m+1} & \cdots & x_{m,n} \\ x_{1,m+1} & \cdots & x_{m,m+1} & x_{m+1,m+1} & \cdots & x_{m+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & \cdots & x_{m,n} & x_{m+1,n} & \cdots & x_{n,n} \end{pmatrix},$$

for any $n \geq m$

Examples on symmetric: 3

Use restriction on 3×3 Cholesky:

$$\left\{ \begin{pmatrix} * & & \\ 0 & * & \\ * & * & * \end{pmatrix} \right\} \text{ acting on } \left\{ \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \in \mathrm{Sym}_3(\mathbb{C}) \right\}$$

Get the free divisor defined by

$$x_{12} \cdot x_{22} \cdot \begin{vmatrix} 0 & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{vmatrix} = 0,$$

which extends to a tower.

Examples on symmetric: 4

Use restriction on 4×4 Cholesky:

$$\left\{ \begin{pmatrix} * & & & \\ 0 & * & & \\ 0 & * & * & \\ 0 & * & * & * \end{pmatrix} \right\} \text{ acting on } \left\{ \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \in \mathrm{Sym}_4(\mathbb{C}) \right\}$$

Get the free divisor defined by

$$x_{13} \cdot x_{23} \cdot \begin{vmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{vmatrix} \cdot \begin{vmatrix} 0 & x_{23} & x_{24} \\ x_{23} & x_{33} & x_{34} \\ x_{24} & x_{34} & x_{44} \end{vmatrix} = 0,$$

which extends to a tower.

Examples on symmetric: 5

(Non-solvable) Consider

$$\left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \text{ acting on } \left\{ \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \in \mathrm{Sym}_3(\mathbb{C}) \right\}$$

Get the free divisor defined by

$$\begin{vmatrix} x_{22} & x_{23} \\ x_{23} & x_{33} \end{vmatrix} \cdot \begin{vmatrix} 0 & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{vmatrix} = 0,$$

which extends to a tower.

Examples on general: 1

Let $\mathrm{GL}_n \times \mathrm{GL}_m$ act on $M(n, m, \mathbb{C})$ by $(A, B) \cdot X = AXB^{-1}$.

Start with $n = m = 1$ and use extension property repeatedly (in a particular way) to get a tower of free divisors on $M(1, 1, \mathbb{C}), M(1, 2, \mathbb{C}), M(2, 2, \mathbb{C}), M(2, 3, \mathbb{C}), M(3, 3, \mathbb{C})$, etc. These give *modified* LU decompositions

Example: On $M(2, 3, \mathbb{C})$,

$$x_{11} \cdot x_{12} \cdot \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \cdot \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} = 0$$

Examples on general: 2

Or, use restriction property to restrict to the subspace where $x_{11} = 0$. Then for $n \geq 2$, using the appropriate group, get a tower, including, e.g.,

$$x_{12} \cdot x_{21} \cdot x_{22} \cdot \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} = 0$$

Examples on general: 3

(Non-solvable) Or, start with a free divisor from [BM06], defined by the product of the maximal minors of a generic $n \times (n+1)$ matrix.

Can expand each of these (in same way) to get a tower including, e.g.,

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \cdot \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} \cdot \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} \cdot \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = 0$$

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Future work

- An arbitrary Lie group is a mixture of reductive and solvable; is there a similar “decomposition” for linear free divisors?
- Why do many free divisors take the form of “determinantal arrangements”?
- Questions?

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- Why do many free divisors take the form of “determinantal arrangements”?
- Questions?

References

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